Supplemental Midterm - Differential Equations (2018-19)

Time: 2.5 hours. Attempt all questions.

1. Let $F : \mathbf{R} \times \mathbf{R}^d \to \mathbf{R}^d$ be a continuous function (we do not assume any Lipschitz condition on F). Assume that $|F(t, \mathbf{x})| \leq M$ when $|\mathbf{x} - \mathbf{c}| \leq K$ and $|t - a| \leq T$. Show that there exists a solution $u : I \to \mathbf{R}^d$ to the differential equation $\partial_t u = F(t, u)$ with $u(a) = \mathbf{c}$, where $I : [a - T_1, a + T_1]$ and $T_1 = \min(T, K/M)$.
(Hint: Without loss of generality assume that $a = 0$ and consider the sequence of functions $u^{(n)}$ given by $u^{(n)}(t) \equiv \mathbf{c}$ for $0 \leq t \leq T_1/n$ and $u^{(n)}(t) = \mathbf{c} + \int_0^{t-T_1/n} F\big(s, u^{(n)}(s)\big)$ for $T_1/n < t \leq T_1$. Use the Arzela-Ascoli theorem. The theorem is recalled below.) **[8 marks]**

2. Find the general solution of $u'' - 2xu' + 3u = 0$. **[5 marks]**

3. Find the general solution of the system

$$\frac{dx}{dt} = 5x + 4y, \qquad \frac{dy}{dt} = -x + y. \quad \textbf{[5 marks]}$$

4. Let $q : \mathbf{R} \to \mathbf{R}$ be a continuously differentiable function such that $q(x) < 0$ for all x. If u is a *nontrivial* solution of $u'' + q(x)u = 0$, then show that u has at most one zero. **[4 marks]**

5. Prove that the critical point $(0, 0)$ of the system $dx/dt = F(x, y)$, $dy/dt = G(x, y)$ is unstable if there exists a function $E(x, y)$ with the following properties: **[4 marks]**

   (a) $E(x, y)$ is continuous and has continuous first partial derivatives in some region containing the origin.

   (b) $E(0, 0) = 0$.

   (c) every circle centered on $(0, 0)$ contains at least one point where $E(x, y)$ is positive.

   (d) $(\partial E/\partial x)F + (\partial E/\partial y)G$ is positive definite.

6. Find the general solution of $u'' + 2u' + 2u = e^{-x}$. **[4 marks]**

Recall:
Arzela Ascoli Theorem: Consider a sequence of real-valued continuous functions $\{f_n\}$ defined on $[a, b]$. If this sequence is uniformly bounded and equicontinuous, then there is a uniformly convergent subsequence.

We say f_n is *uniformly bounded* if there is a $M > 0$ such that $|f_n(x)| \leq M$ for all n and x.
We say f_n is *equicontinuous* when for every $\epsilon > 0$ there is a $\delta > 0$, such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$ for all n.
We say f_n is *uniformly convergent* to f if $\sup_{x \in [a,b]} |f_n(x) - f(x)| \to 0$.